Through the city of Königsberg in East Prussia there flows a river containing two islands within the city limits. These islands are connected with each other and with the river banks by seven bridges as shown in Fig. 1. A little over

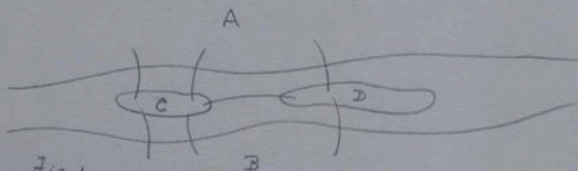


Fig. 1.

two hundred years ago the citizens of the city carried on a lively debate regarding the question of whether it was possible to take a walk along a path which should pass over each of the seven bridges once and just once. After many repeated attempts to do so no one had succeeded. This was, of course, no proof that a path could not be found. In 1735 the great mathematician Euler heard of the problem of the citizens of Königsberg. He immortalized the problem by solving it and at the same time created a branch of the field of mathematics known as topology.

In truly mathematical fashion he idealized the problem by abstracting from it its essential mathematical character. Since there is no re-

striction on the path so long as it remains
If the bridges be replaced the river banks
by two points A and B and the two islands
by two points C and D. Thus idealized Fig. 1
becomes Fig. 2,

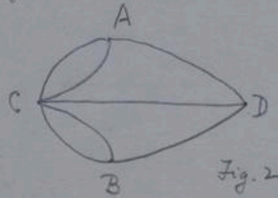


Fig. 2

where the arcs joining the points represent the
bridges. The problem now reads as follows:
<u>Can the complete figure above be traced by
a single continuous line without passing
over any arc more than once</u>?

Since there are only seven arcs and four
vertices it is clear that one could make
a catalog of all possible paths, and were
one to do so he would find that no path
of the kind required would occur in the list.
This would be a solution of the problem.
But it would be unsatisfactory from a mathe-
matical standpoint because it would not
show why the answer to the question is
negative.

In solving the problem Euler showed why no path could be found and in addition showed how one could determine in similar though more complicated problems whether a path could be found. Euler's solution is not particularly difficult but it is not quite so interesting as later solutions, one of which I shall now give.

It is, of course, clear that any admissible path must contain the vertex A. Thus there are three possibilities: A is either the beginning or the end of the path, or it is neither. I shall show that the third possibility never occurs by proving the following statement.

α) If A is not the beginning of an admissible path then it must be the end of the path. We suppose then that A is not the beginning of the path. The path must then come to A for a first time by some arc. There then remain two arcs issuing from A not yet traced. Hence the path must leave A on one of these. There is now one arc issuing from A not yet traced. By hypothesis the path must trace this third arc, and it is clear it must trace it coming toward A. After this arc is traced the path cannot obviously leave A again. Hence A must be the end of the path.

The essential fact upon the basis of which we proved statement α) is that there is an odd number of arcs issuing from A. We now note that this is true also for B, C and D. Thus we conclude at once that any admissible path must have all four of the vertices A, B, C and D either as beginnings or as ends. But a single continuous curve has only one beginning and one end, and we accordingly conclude that the answer to the question is negative: <u>there is no single continuous curve tracing Fig. 2 which passes over each arc just once</u>.

If we call a vertex <u>odd</u> when the number of paths issuing from it is odd, and otherwise call it <u>even</u>, we see that the reason why the answer to our question is negative is that there are too many odd vertices in the figure. In fact we see more, namely, not more than two odd vertices can be allowed.

Suppose we now alter the problem slightly by inserting an additional arc between A and B, thus obtaining Fig. 3. We note that now A and B

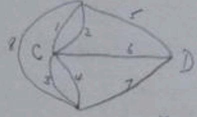


Fig. 3.

are even vertices, C and D still odd. And we see also that the path C 1 A 2 C 3 B 4 C 6 D 5 A 8 B 7 D is a single continuous curve which traces the complete figure and passes over each arc just once.

Obviously the remarks made above can be applied to the general case in which there are n vertices joined in pairs by m non-intersecting arcs. Such a collection of vertices and arcs is called a <u>linear graph</u>. If any two vertices of the graph can be joined by a succession of arcs of the graph, the graph is said to be <u>connected</u>. Reviewing the argument above we can at once state the following theorem.

I. <u>If a connected linear graph contains more than two odd vertices there is no single continuous curve which traces the complete graph and which passes over each arc just once.</u>

But we must be careful to notice at once that we have <u>not</u> proved the following companion theorem.

<u>II. If a connected linear graph contains at most two odd vertices there exists at least one single continuous curve which traces the complete graph and which passes over each arc just once.</u>

We noted in the case of the particular graph in Fig. 3 that this could be done. And it is true that the theorem can be proved. But any proof of it is considerably more sophisticated than the proof of the former theorem. A consideration of the figures below shows that the complexities which are possible are many.

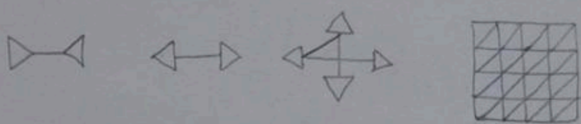


Fig. 4.

Before proving theorem II we make a definition and prove another theorem. By an <u>Euler graph</u> we mean a linear graph all of whose vertices are even. The theorem is the following.

<u>III. A connected Euler graph can always be traced by a single continuous closed path or curve which passes over each arc just once.</u>

The proof of theorem III rests on the following statement.

(β) <u>In an Euler graph each vertex is contained in a closed path which traces any given arc at most once.</u>

Let P be any vertex of an Euler graph. Starting from P we trace an arc to a vertex say P_1, from P_1 to P_2, etc, thus obtaining a succession $P P_1 P_2 P_3$ ... We agree never to trace the same arc twice. This is always possible until we return to P, for each vertex is even. And I say that we must return to P, for there is only a finite number of arcs and vertices. Hence, upon returning to P we have traced the closed path demanded in the statement (β).

We can now prove theorem III. Let us call the given connected Euler graph G. Since the number of arcs in G is n it is clear that no closed path in G which traces each arc at most once can contain more than n arcs. Let the greatest number of arcs contained in any such closed path be l. Then $l \leq n$. Let Z be a closed path containing l arcs. If we can show that $l = n$ then Z must contain every arc of G and we shall have proved theorem III. Suppose then that Z does not contain every arc of G. Let the remaining arcs of G, together with their vertices be called G'. Then G' is clearly an Euler graph, i.e. each of its vertices is even. Moreover G and G' must have at least one vertex, say Q, in common. For otherwise, no vertex of G' can be joined by a succession of arcs of G to any vertex of Z. But this is a contradiction of the fact that G is connected.

Since G' is an Euler graph it follows from β) that there is in it a closed path Z' which contains Q and which traces any given arc of G' at most once. Thinking of Z' as beginning and ending at Q we can now start at Q, trace Z in G, thus returning to Q, and then trace Z' in G', returning finally to Q. The total path Z + Z', which is clearly closed and traces no arc more than once, must thus contain more than l arcs. This is a contradiction, since l is by hypothesis the greatest number of arcs in any such closed path. Accordingly we now conclude that Z does contain every arc of G, and the proof of theorem III is complete.

We return now to the proof of theorem II. We are supposing that the graph G contains at most two odd vertices. If it contains none, theorem II follows at once from theorem III. If G contains one odd vertex A it must contain a second one B. For since each arc has two ends the total number of ends of arcs is even, and this is exactly the total number of arcs issuing from the vertices, each original arc of G being counted twice. If A were the only odd vertex this total number would be odd and not even, as it must be.

We now join A and B by a new arc γ. In the new graph G' thus obtained A and B are even vertices. Hence by theorem III G' can be traced in the required manner by a path Z. If we think of Z as starting at A and tracing G' with γ as the last arc traced we see that Z less the arc γ traces G in the required fashion. Hence the proof of theorem II is complete.

However we observe that neither theorem II nor theorem III gives us a method for tracing a given graph. And a study of a few simple examples will show that one may still have difficulty in finding a proper path even when one is known to exist.

The problem of the Konigsberg bridges and its generalization discussed above are notably different in two respects from the one we shall now look at. In the first place it is essentially purely a combinatorial problem involving a finite number of objects, the vertices and the arcs joining them; there is no question of where these objects are, whether they are on a plane, on a sphere, in three-space, etc. In the second place, the figure to be studied is given and does not need to be constructed in the sense at least that we shall see below. The problem is further distinguished by the fact that both in its statement and its solution it involves concepts readily understood in all their generality by an intelligent grammar school pupil.

The problem, or puzzle, that we turn to next is readily understandable in its statement, in fact many a person can remember having become acquainted with it at a very early age. Its complete solution, however, involves a very sophisticated theorem in topology. The problem is as follows. Let A, B, and C be three houses and let 1, 2 and 3 be three wells. It is required to join each house to each well by pipes lying on the surface of the ground and such that no two pipes cross each other. The statement of the problem is simplicity itself.

It should be noted at once that the pipes are to lie on the surface of the ground, that is, on the surface of the earth. The problem thus involves not only the houses, wells and pipes, but also the surface on which these are to be considered. It should further be noticed that while in the Königsberg problem a finite number of paths only could occur, here an infinite number may occur, – there is an infinite number of possible paths from each house to each well and any one of these may be of extreme complexity. Thus in a sense it is manifestly impossible to solve the problem by testing all possibilities. I leave it to you to attempt a few, thus convincing yourself of the complexity of the situation, and I may remark that you will not find any scheme satisfying the required conditions.

Before proving the impossibility of solving the above problem let us change it in the following respect. Instead of stating the problem for the earth, which is essentially a sphere, let us suppose that the earth is essentially a torus, which is the mathematical name for a doughnut (with a hole!). Here one sees at once that the solution is feasible, as indicated in Fig. 5, where the dotted line

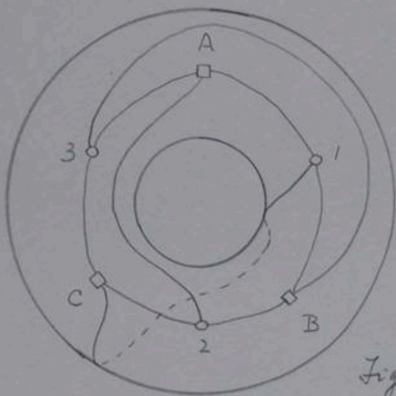


Fig. 5.

is on the underside of the torus. We should note at once that in this solution at least one of the pipes goes through the hole of the torus. There are many other solutions, but in every one of them, one of the pipes must go through the hole. For if this were not so we could fill in the hole with a double sheet of rubber, and have essentially a sphere, so far as the problem in hand is concerned. But then we would have the problem solved on the sphere, and as we shall see this is impossible.

It is time now to inquire exactly what is the difference between a sphere and a torus which makes the problem of the houses and wells unsolvable in the first case and solvable in the second. A first attempt at stating the difference is to say that a torus has a hole while a sphere doesn't. But this is not very illuminating nor does it really help us to see why the possibility of solution differs in the two cases. There are many technical ways of stating the difference between a sphere and a torus. Perhaps the most intuitively clear one and one which is immediately of use in discussing the problem in hand rests on the so-called Jordan curve theorem.

Before stating the theorem we must make a definition. A <u>simple closed curve</u> on a surface is any closed curve curve on the surface which does not intersect itself. Intuitively it is any curve which can be obtained from a circle on the surface by continuously deforming the circle on the surface, stretching it or contracting it but not breaking it. The Jordan curve theorem is the following:

IV. <u>Every simple closed curve</u> γ <u>on the surface of a sphere divides the surface of the sphere into two regions R and S each of which has γ for its total boundary, and such that no point of R, not on the boundary of R, can be reached by a path on the surface from any point of S not on the boundary of S, unless the path crosses</u> γ.

Only a mathematician or an idiot would see anything in this theorem which he did not think perfectly obvious. However, really to prove this 'obvious' fact requires very high-powered mathematical machinery, and we shall accordingly not attempt to prove it.

The important thing for us is that the

theorem is not true of it if we replace the word 'sphere' by the word 'torus'. Some closed curves, for example k in Fig. 5, do divide the surface of the torus into two regions, but some do not, for example l in Fig 6.

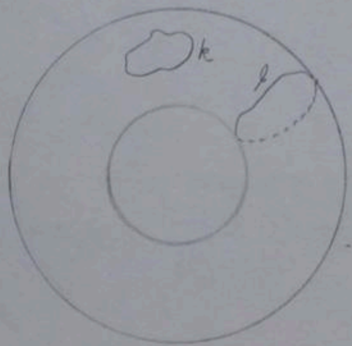


Fig. 6.

Exactly what now is the bearing of the Jordan curve theorem on the 'houses and wells' problem on the sphere? We can see this in the following way. We begin with the houses A, B, C and the wells 1, 2, 3 placed at random on the sphere. If the problem can be solved there must exist the following path of pipes, namely A 1 B 2 C 3 A, as in Fig. 7. This path is a simple closed curve which, by the Jordan theorem, divides the surface of the sphere into two regions, α with boundary

A 1 B 2 C 3 A and β with boundary A 3 C 2 B 1 A. We note the boundaries are written in different orders, namely, in each case from left to right as one stands inside the region.

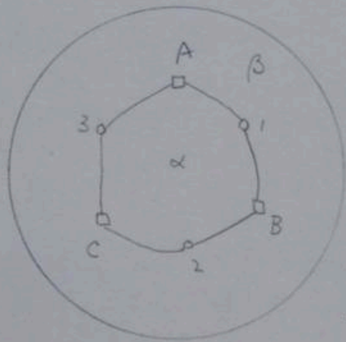


Fig. 7

In a very real sense α and β are indistinguishable, each of them is 'inside' the path A 1 B 2 C 3 A and each of them is 'outside' of it.

Further, if the problem can be solved there must be a pipe from A to 2, and this pipe must lie wholly in α or wholly in β. Which we choose is clearly immaterial. Let us choose α, as in Fig 8. We now need another theorem equally as obvious as the Jordan curve theorem, namely the following.

<u>A simple non-self-intersecting arc joining two points of the boundary of a region</u>

whose boundary is a simple closed curve and which itself lies wholly in the region divides the region into two parts such that no interior point of either can be reached from an interior point of the other by a path in the original region without crossing the arc.

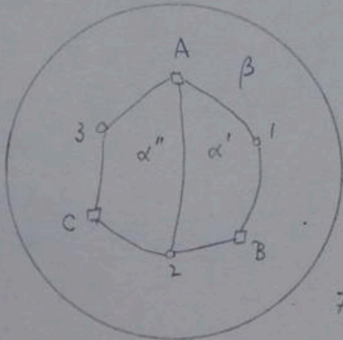


Fig. 8

Thus we see that the arc A2 divides α into two regions, α' with boundary A 1 B 2 A and α" with boundary A 2 C 3 A.

Proceeding further, if the problem can be solved there must be a pipe from B to 3. Now B and 3 are on the boundary of both α and β, and the pipe B3 must lie wholly in α or in β. But B is on the boundary of α' and 3 is on the boundary of α". We thus see, by a simple extension of the above theorem that B3 cannot lie in α, for it would have to cross A2. But it can lie in β for β has not been divided. Let B then be joined to 3 in β, as in Fig. 9. The arc B3 will divide β into

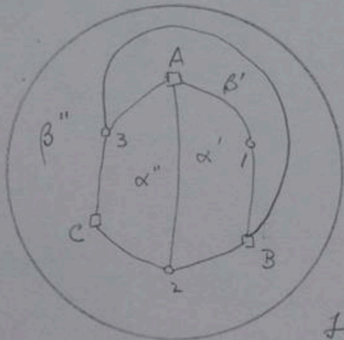


Fig. 9

two regions β' with boundary A3B1A and β'' with boundary C2B3C.

There remains only one pipe to be laid, namely C1. It is impossible to lay this pipe. We have only to show that some closed path of pipes already laid divides the surface of the sphere into two regions, one containing C in its interior and the other containing 1 in its interior. I say this curve is A2B3A, and that the regions are $\alpha' + \beta'$ and $\alpha'' + \beta''$.

Certainly the curve A 2 B 3 A is closed. Further as one easily sees by considering the way in which the boundaries have been written down, α' and β' adjoin, along A 1 B, and the boundary of $\alpha' + \beta'$ is A 3 B 2 A, while α'' and β'' adjoin, along 3 C 2 and the boundary of $\alpha'' + \beta''$ is A 2 B 3 C.

We thus see that the 'houses and wells' problem cannot be solved on the sphere. We have shown by an example that it can be solved on the surface of a torus. To show exactly where the Jordan curve theorem bears on the latter problem would take us too far. We may however point out that on the double torus, a doughnut with two holes, the problem can be solved even when the number of houses and the number of wells are each increased to four. The solution is shown on page 20. It is a reasonable guess, though very possibly incorrect, that on a torus of n holes the problem can be solved when the number of houses and the number of wells are each $n+2$. We have seen the statement to be true for $n = 0, 1, 2$.

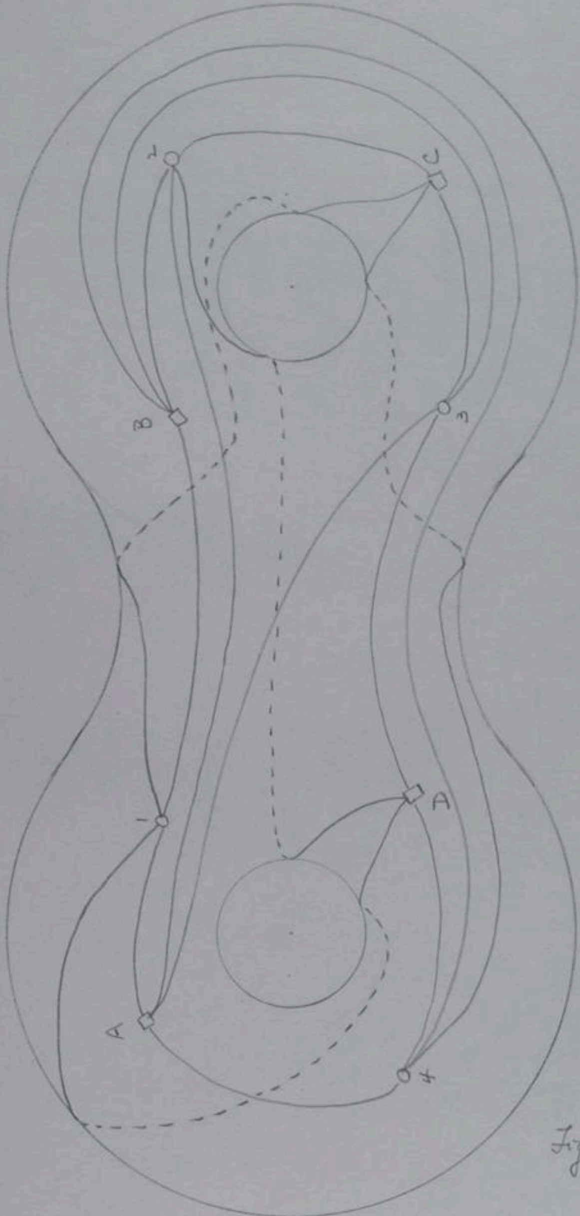


Fig. 10

Another similar problem which I recommend as interesting is the following.

Given a torus with n holes. What is the largest number of points $P_1, P_2, P_3, \ldots, P_r$ which can be taken on the surface such that each point can be joined by one arc to each other point with none of the arcs crossing each other?

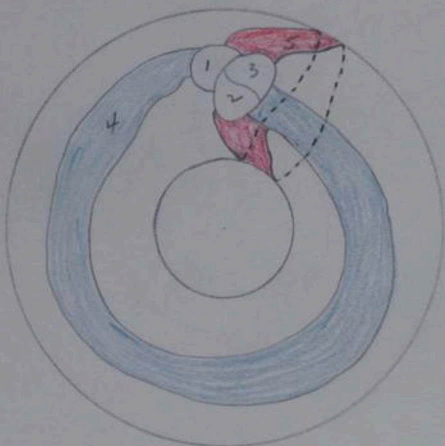
Still another problem is suggested by the following story. A king who had five sons provided for them in his will by saying that each should receive a fifth of his kingdom if they could so divide it into five parts in such a way that each two parts should have a common boundary. It was not necessary to record that all of them had died before finding how to subdivide the kingdom, for the subdivision is impossible.

To prove this statement we shall content ourselves by generalising the problem and stating the present stage of its solution. The generalization which is of interest to us is known as the Four Color Map Problem. Does there exist a map on the surface of the sphere which cannot be colored in four colors, in such a way that no two regions which have a common boundary have the same color?

Despite the apparent simplicity of this problem its complete solution is unknown. It is known however that every map on a sphere of 35 or less simply connected regions can be colored in four colors. A simply connected region is one which can be continuously deformed on the surface into a circular disk.]

Using the above result we can now see why the problem inherited by the sons of the king was not solvable. For if it were we should have a map of five regions requiring five colors, since every region must border each of the remaining four.

Again we note the difference between the sphere and the torus. If the kingdom covered a sufficiently complicated area of the torus it could be divided as required. See Fig. 11.



It is thus clear that on a torus at least five colors may be necessary to color a map. Actually there are maps which require 7 colors.

Before we close let us look at another technical distinction between the sphere and the torus. Let $P_1, P_2, \ldots, P_V$ be V points on the surface of the sphere. Let $A_1, A_2, \ldots, A_E$ be E arcs joining various pairs of these points, the only restrictions being the following: 1) every point P_i shall be an end of at least two arcs A_j; 2) no two arcs shall intersect except at their endpoints; 3) the regions into which the arcs cut the surface of the sphere shall be bounded by simple closed curves. The first two restrictions are clear in nature. To understand the third consider Fig. 12. The stippled region is bounded

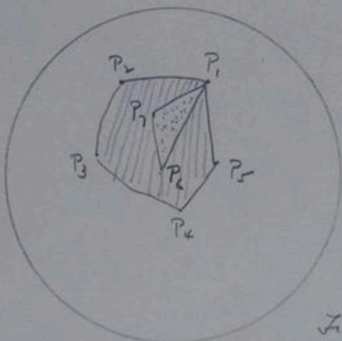


Fig. 12

by a simple closed curve, but the boundary of the shaded region has a double point, P_1. By inserting another arc, for example, $P_2 P_7$, the shaded area will be cut into two regions of the required character.

We shall call anyone of the simple regions into which the surface of the sphere is thus divided a face and let us suppose there are F faces. And we shall call the arcs, edges, and the points, vertices. Then the following relation, known as Euler's Formula, holds:

$$F - E + V = 2$$

The proof of this relation is very simple. Let R_1 be one of the faces, bounded, say, by $P_1 E_1 P_2 E_2 \cdots P_h E_h P_1$. All of the remaining faces may now be thought of as forming a single face R having the same boundary as R_1. Thus for the moment there are two faces R_1 and R_1 h edges, $E_1, \ldots, E_h$ and h vertices, $P_1, \ldots, P_h$. Clearly then in this case $F - E + V = 2 - h + h = 2$. If the original subdivision contained more than one face, the 'face' R must contain a second face R_2, which we can clearly suppose adjoins R_1 along at least one edge and is such that its common boundary with R_1 is a connected sequence of edges.

The part of its boundary which is not on the boundary of R_1 will have one more edge than vertex. Hence we have increased E one more than V. And we have clearly increased the total number of faces from 2 to 3. Thus still the relation $F - E + V = 2$ holds.
It is now clear that proceeding in this manner all of the original subdivision may be obtained by successive steps, at each stage of which the Euler Formula holds.

What now is the corresponding situation for the torus? If we make the same sort of subdivision on the surface of a torus and add one additional restriction the formula becomes

$$F - E + V = 0.$$

The additional restriction is the following: 4) Every region must be continuously deformable on the surface into a circular disk.

The proof in this case is equally simple and I suggest that you try it. In the case of a torus with with p holes the formula becomes

$$F - E + V = 2 - 2p.$$

The number $2-2p$ is called the <u>Euler-Poincaré characteristic</u> of the surface.

Finally it is worth noting in the case of the sphere and torus that $2p$, which is

0 for the sphere and 2 for the torus, is equal to the number of essentially distinct closed curves which can be drawn on the surface without separating it. The two possible types of curves for the torus are shown in Fig. 13 as k and l.

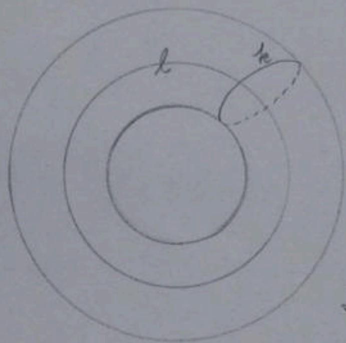


Fig. 13.

A List of References.

Königsberg Bridge Problem

Lucas, E., Récréations Mathématiques. V.1, pp 21-38
(All the four volumes of this work contain many interesting problems, both topological and otherwise).

Four Color Map Problem.

Lucas, E., Récréations Mathématiques, V.4, pp. 168-188.

Franklin, Philip, The four color problem, Scripta Mathematica, V.6, 1939, pp. 149-156, 197-210.

Franklin, Philip, The four color problem, Long Island University. Galois Institute of Mathematics Lectures, #13.
(The latter, which is a volume of mimeographed lectures, contains many papers of a simple nature, treating topological problems. It is in our library).

Elementary Topology

In addition to the works cited above there are the following.

Kasner, E. and Newman, J., Mathematics and the Imagination, pp. 265-298

Hilbert, D. and Cohn-Vossen, S., Anschauliche Geometrie.
(neither of the latter books is in our library, but it is hoped to add them soon).

A List of References

Königsberg Bridge Problem

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Four Color Map Problem

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Elementary Topology

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Kasner, E. and Newman, J., Mathematics and the Imagination, pp. 265-298, New York, Simon and Schuster, 1940

Hilbert, D. and Cohn-Vossen, S., Anschauliche Geometrie, Berlin, Springer, 1932

The first four references can be found in our library. It is hoped soon to add the last two.

Outline for Math Club talk, Nov. 19, 1940.

I. Königsberg bridge problem
   Euler, 1735
   Catalog of paths (645, 120)
      Unsatisfactory.
   Euler's solution.
   Our solution.
      Statement α)
      Appl. to ABCD.
      No solution.
   Odd & even vertices.
      Reason for no solution.
      Case in which there is a solution.
   Linear graph, connected.
   Th. I.
   Th. II
      Diff. of proof illustrated by examples. Fig.
   Euler graph.
   Th. III.
      Statement β) Fig
      Proof of III. Fig
   Proof of Th. II.
   Finding a path.

II. Character of K. problem as distinguished from next problem
   Problem of houses & wells.
      Importance of surface
      Catalog not feasible.
   Problem on the torus. Fig
      Note pipe through hole
   Question of diff. between sphere & torus.
   Simple closed curve
   Jordan curve Th.
      False on torus.

appl of J. C. Th. to show no solution on sphere Fig. (3)

Case of double torus; torus with n holes.

Recommended problem.

III. Problem of subdividing the kingdom.

Four color map Problem. Fig

Known solution.

Appl to Kingdom problem

Case on torus Fig

IV. Euler's Th. Fig

Proof Fig

Case on torus

Torus with p holes.

$2p$ = closed curves.

$-E + V = -N$

$-1 + 2 = +1$

$-2 + 3 = +1$

$-3 + 3 = 0$

$-4 + 4 = 0$

$-5 + 4 = -1$

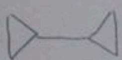
$-6 + 5 = -1$

$-2 + 2 = 0$

$-7 + 5 = -2$

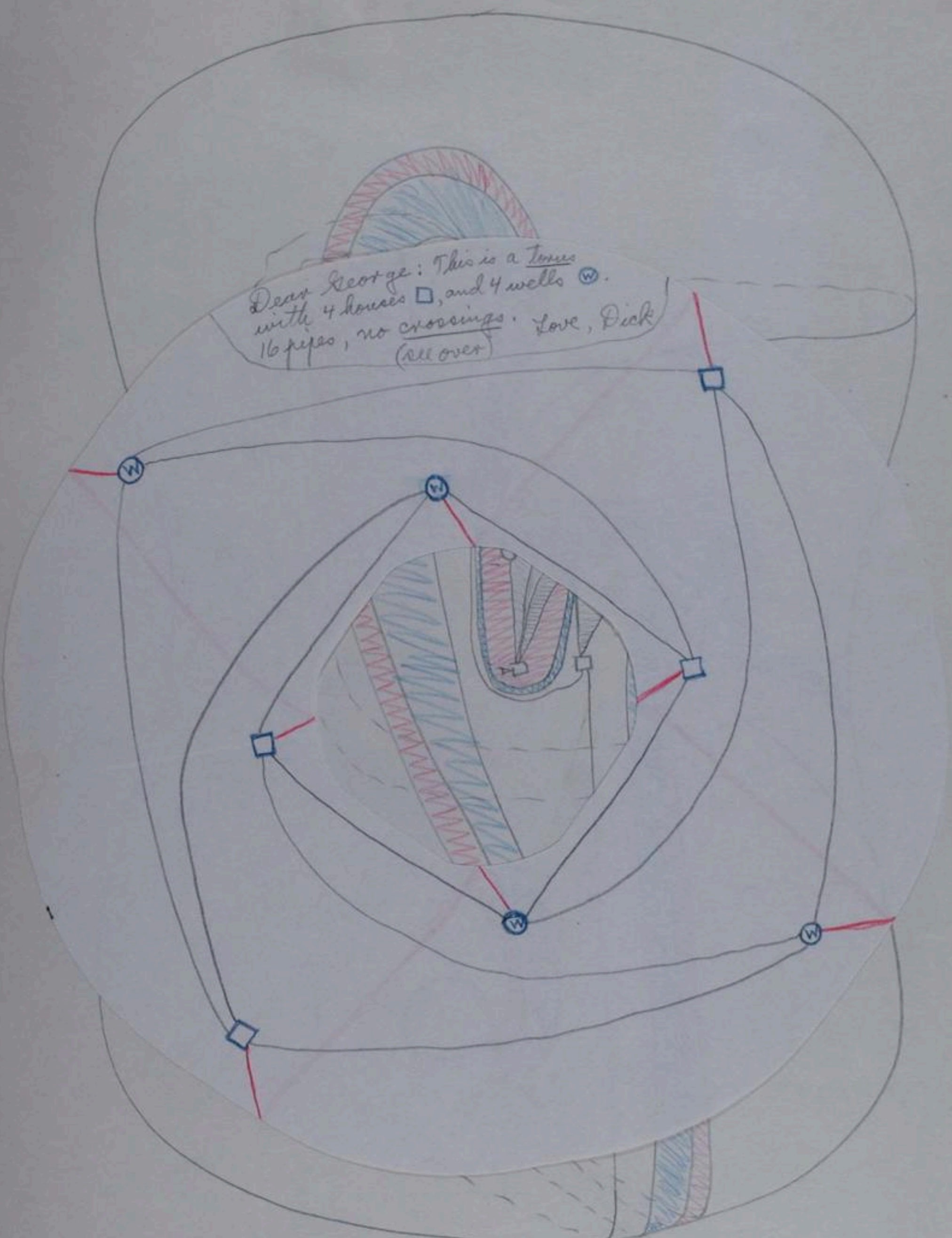
$-12 + 7 = -5$

$-12 + 8 = -4$



$-7 + 6 = -1$

$-7 + 7 = 0$


Dear George: This is a torus
with 4 houses □, and 4 wells Ⓦ.
16 pipes, no crossings. Love, Dick
(see over)
W
W
W
W

Given n points on torus with p holes. Join each point to each other point by one and only one simple arc. Find maximum value of n if all regions of subdivision are 2-cells having simple closed curve triangle boundaries

Let x be the number of $\triangle$.

Now $\frac{n(n-1)}{2}$ = number of arcs.

Also number of arcs $= \frac{3x}{2}$

Thus

$$3x = n(n-1)$$

By Euler's formula

$$x - \frac{n(n-1)}{2} + n = 2-2p; \quad -\frac{x}{2} + n = 2-2p$$

$$3x - 6n = -12+12p$$

$$n(n-1) = 6n - 12 + 12p$$

$$n^2 - 7n + (12 - 12p) = 0$$

$$n = \frac{7 \pm \sqrt{49 - 48 + 48p}}{2}$$

$$= \frac{7 \pm \sqrt{1+48p}}{2}$$

$$1 + 48p = q^2.$$

$$p = \frac{q^2-1}{48}$$

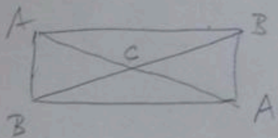
$q = 1, 7, 17, 31, 49$ at least

$49^2 - 1 = (49-1)(49+1), \quad p = 50$

	p	n		x	
$q=1$	0	4	3	4	2
$q=7$	1	7	0	14	
$q=17$	6	12		44	
$q=31$	20	19		114	
[illegible]	[illegible]	28		252	

	8	q^2	q^2-1
1×	8	1	0
2×	6	4	3
3×	4	9	8
4×	2	16	5
5×	0	25	4
6×	8	36	5
7×	6	49	8
8×	4	64	3
9×	2	81	0

Mobius Strip



$F - E + V = 0$

$4 - 7 + 3 = 0$

Proj Plane $\quad F - E + V = 1$ (By adding disk to Mob. strip)

Disc

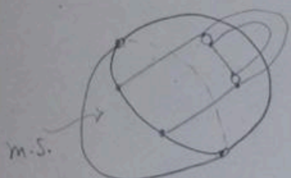
$F - E + V = 1$

$2 - 3 + 2 = 1$

Proj Plane with one handle

$F - E + V = -1$

F	E	V
4	11	6



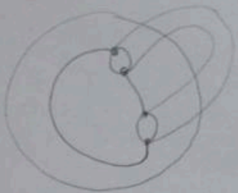
Proj. Pl. with 2 handles $\quad F - E + V = -3$

p handles $\quad F - E + V = 2 - 2p - 1 = \underline{1 - 2p}$.

Sphere $F - E + V = 2$ $p = 0$ $2 = 2(1-p)$

Torus $F - E + V = 0$ ✓ $p = 1$ $0 = 2(1-p)$

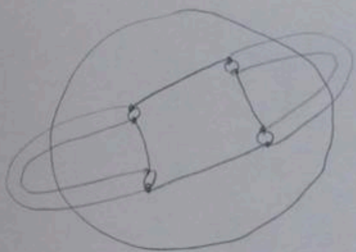
$4 - 8 + 4 = 0$



Double Torus

$F - E + V = -2$ $p = 2,\ -2 = 2(1-p)$

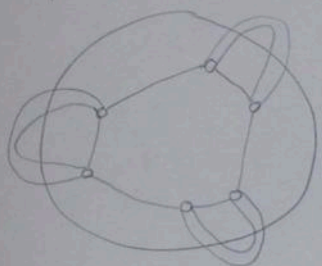
$6 - 16 + 8 = -2$



Triple Torus

$F - E + V = -4$

$8 - 24 + 12 = -4$



Sphere with p handles. gen'l. $F - E + V = 2 - 2p$.

Graphs.

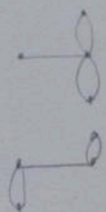
$-E + V = -p_1 + p_0 = -\varepsilon$ $\qquad$ $\varepsilon = E - V$

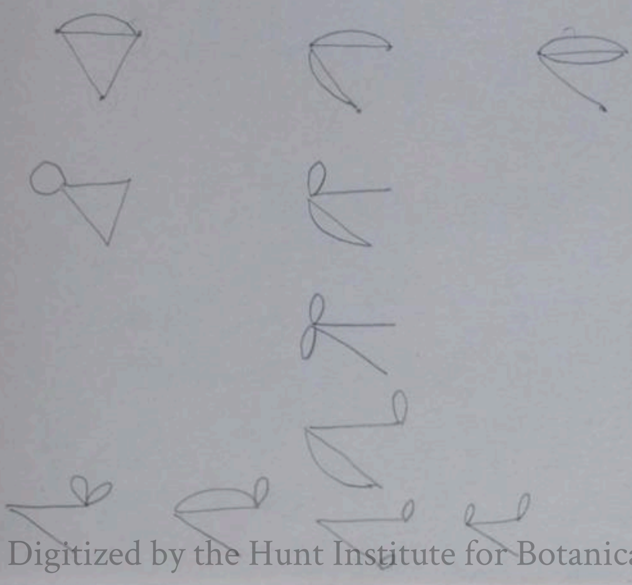
$\varepsilon = 1$

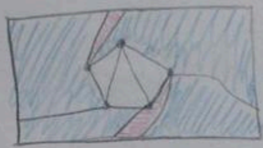
$p_1 = \varepsilon + p_0 = 2$

$\varepsilon = 1$

$p_1 = 2$







Torus

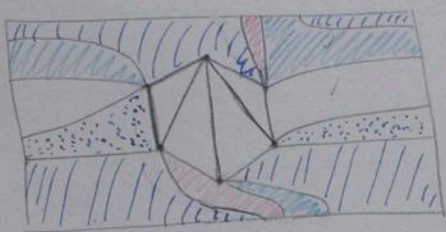
$F - E + V = 0$

$5 - 10 + 5 = 0$?

no

$5 - 9 + 5 = 1$

indicates a non-simply conn. reg.
(the blue)

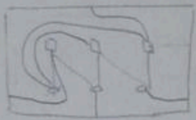


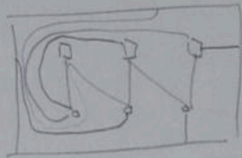
Torus

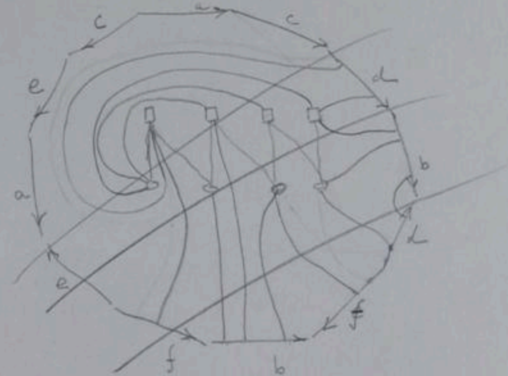
✓

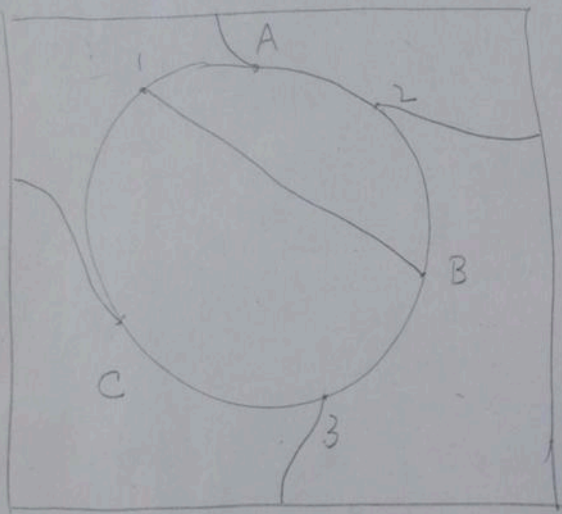
$9 - 15 + 6 = 0$

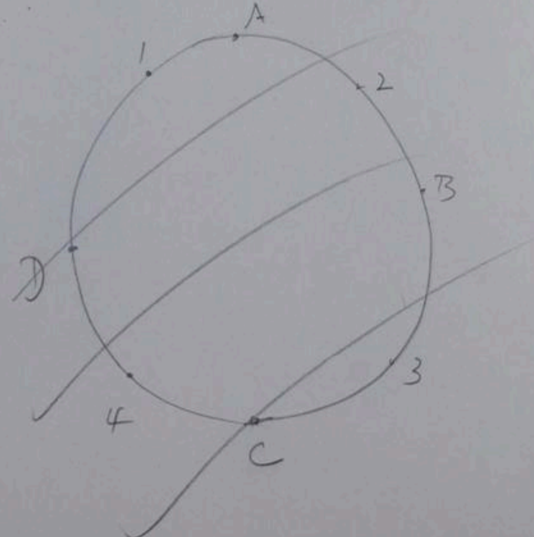
?∴ no non-simply conn.







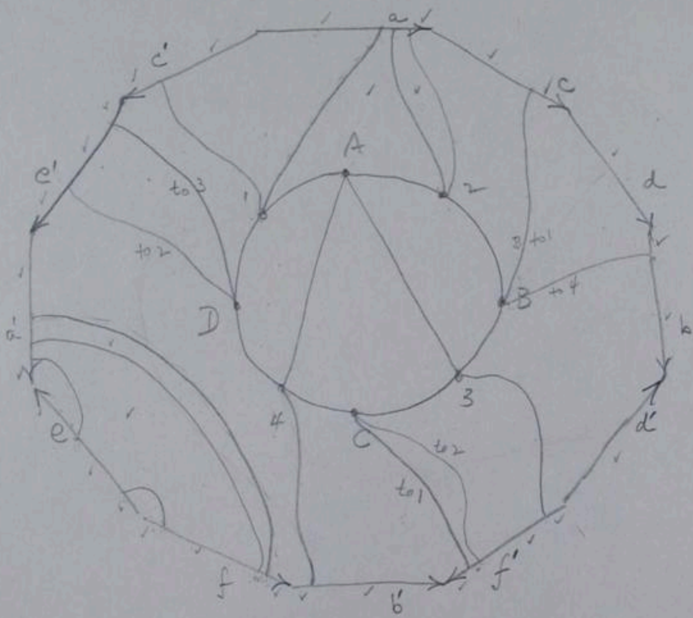




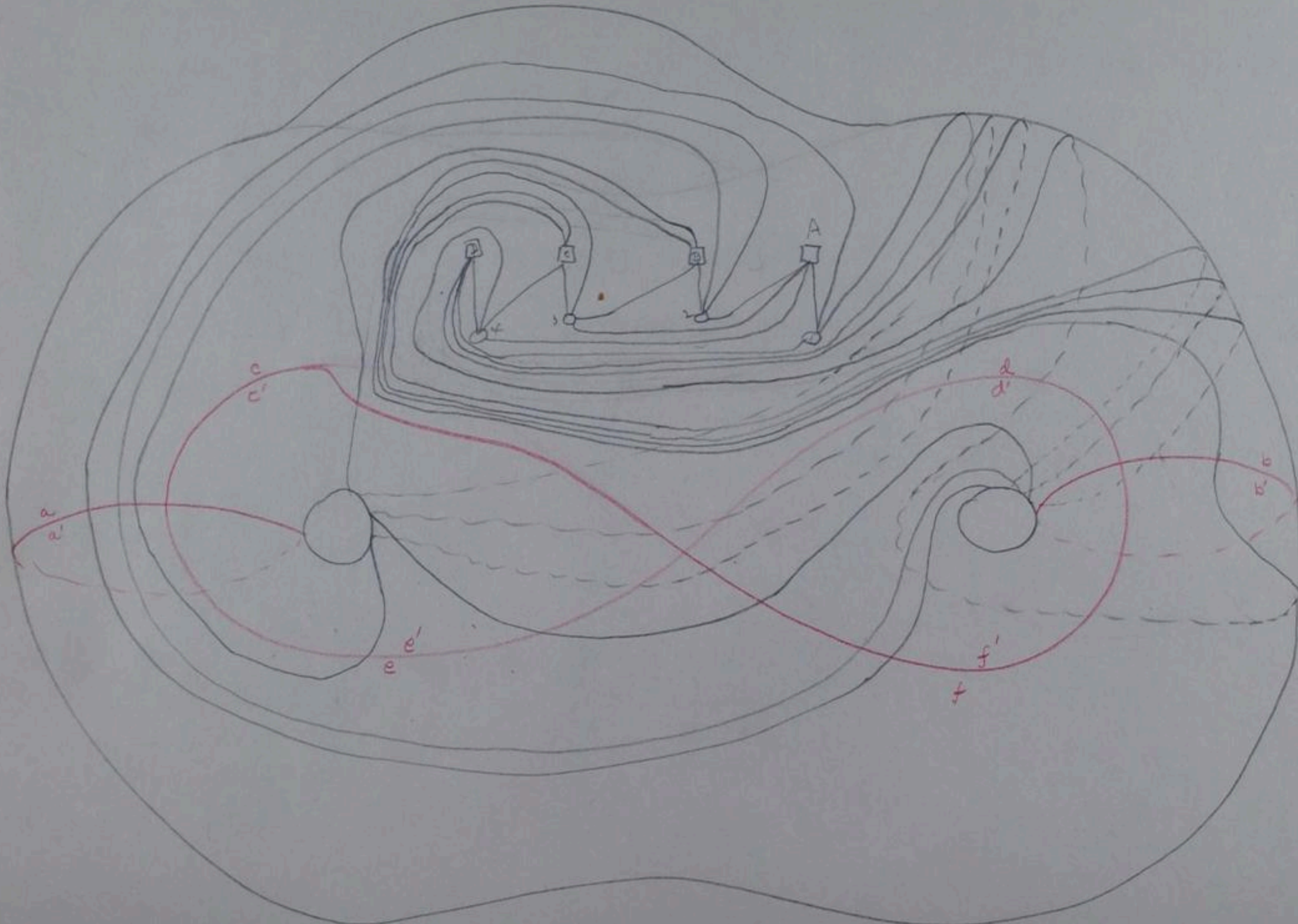
$6 - 15 + 9 = 0$ ✓

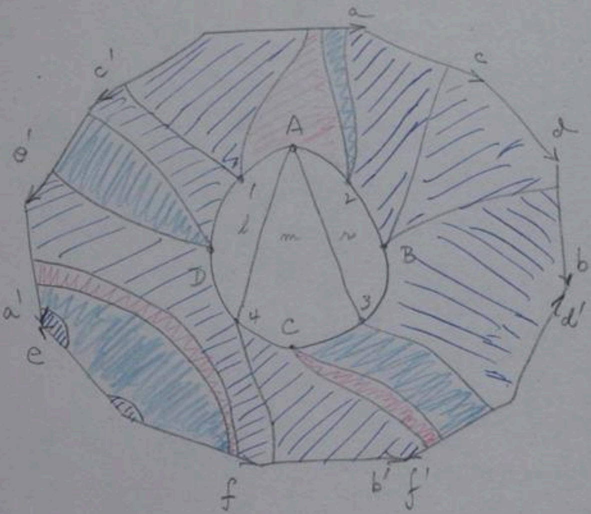
F E V

$19 - 42 + 21 = -2$



Four Houses + Four Wells on Double Torus.


A
a
a'
b
b'
c
c'
d
d'
e
e'
f'
f



Boundaries: A 1 D 4 A

A 4 C 3 A

A 3 B 2 A

2 D 3 C 2

A 2 C 1 A

2 B 3 D 1 C 4 D 2

F = 6

E = 14

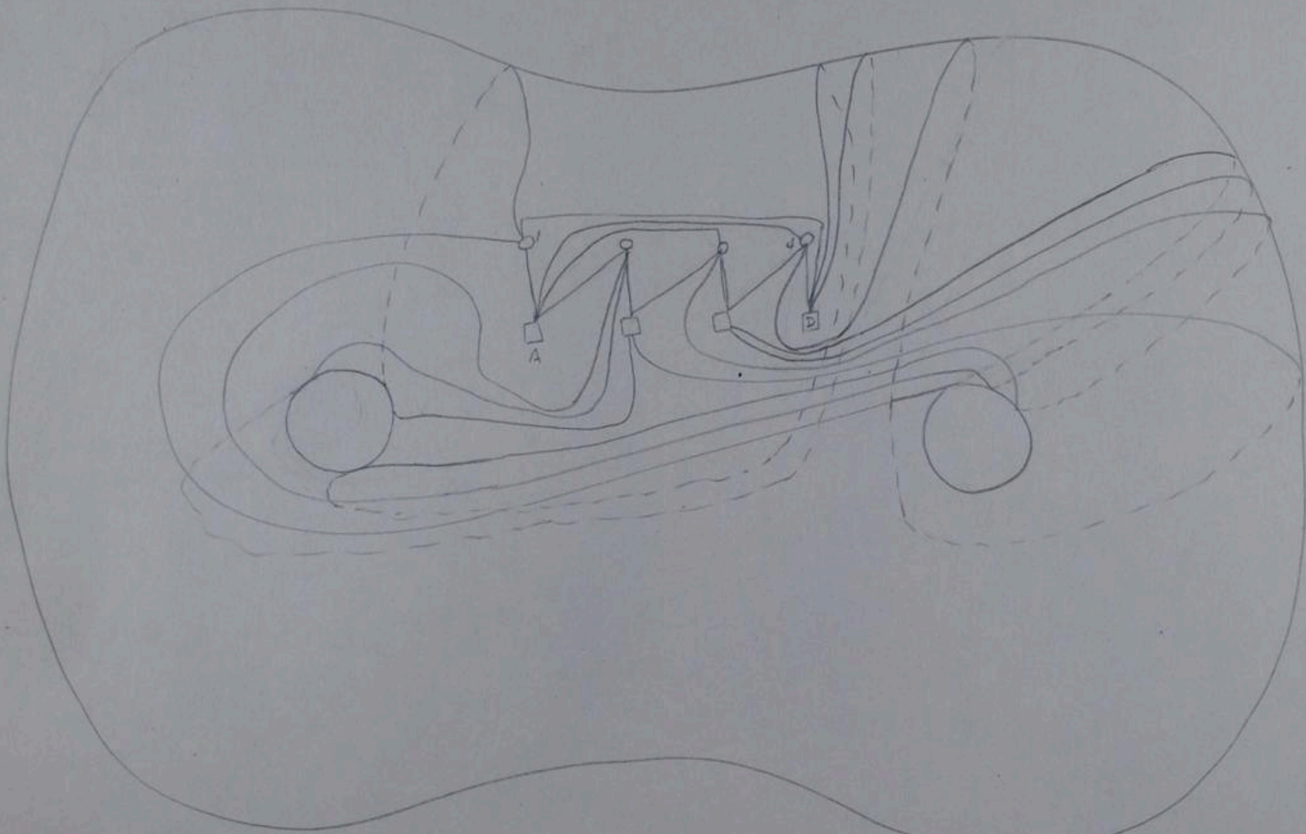
V = 8

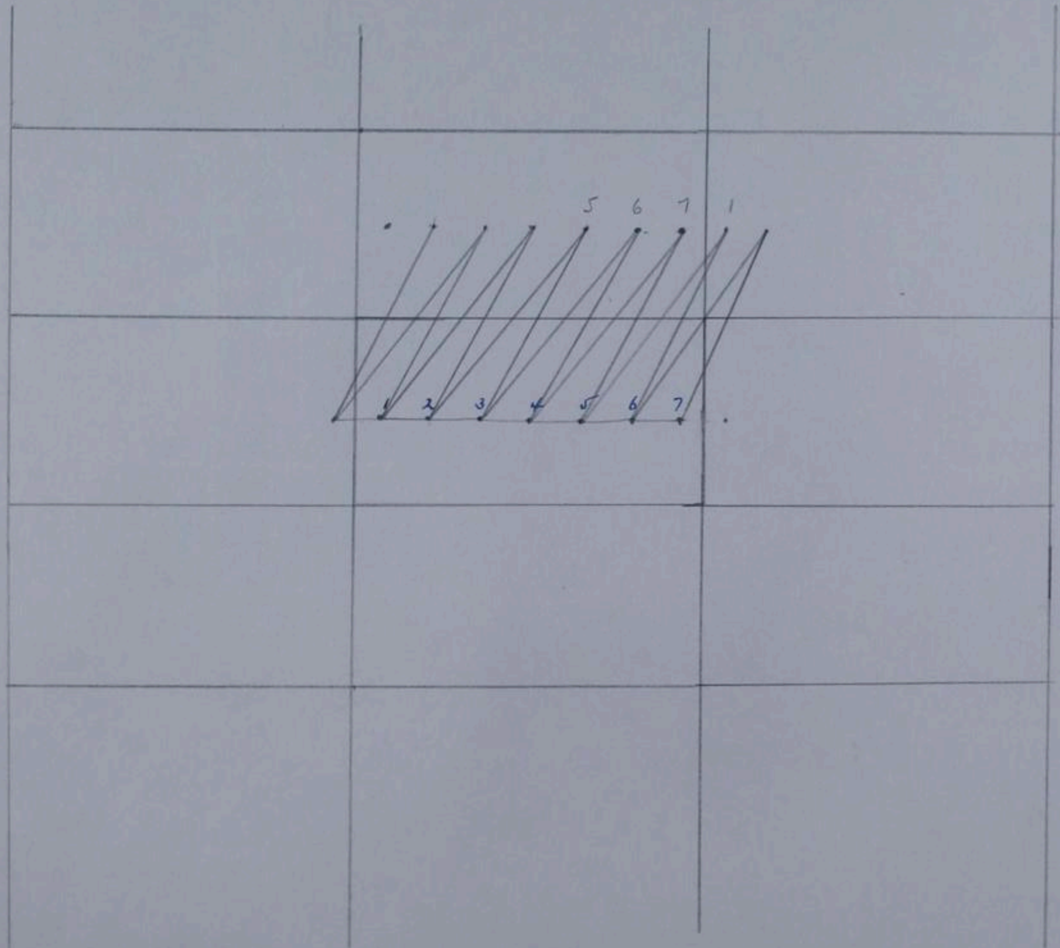
6 − 14 + 8 = 0. indicating a non-simply connected region.

l touches red, ink, m
m " l, r, ink, blue
r " red, ink, m
red " l, r, ink, blue
blue " red, ink, m
ink " l, r, m, red, blue

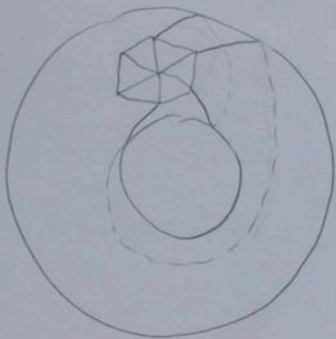
Since blue doesn't touch l or r can make l or r blue & use only 4 colors namely red blue ink m

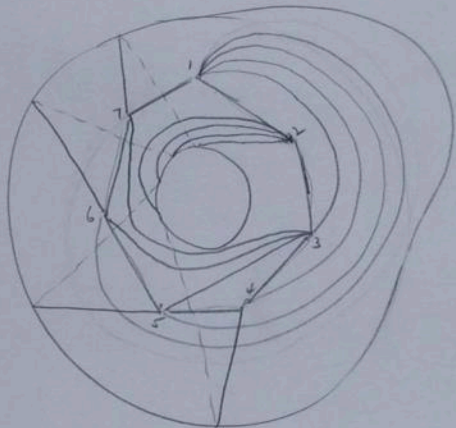
In fact can use 3 colors by further making m = red


A
D


5 6 7 1
1 2 3 4 5 6 7

8 V	6 V	6
28 E	15 E	15
$\frac{56}{3}$ F	$\frac{30}{2}$ F	15





12
13
14
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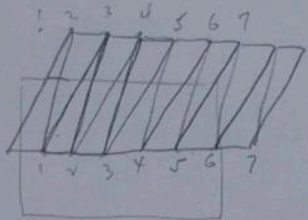
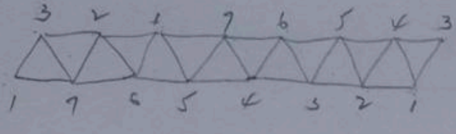
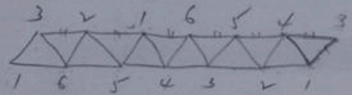
~~34~~
~~35~~
~~36~~
~~37~~
~~45~~
46
47
~~56~~
57
~~67~~



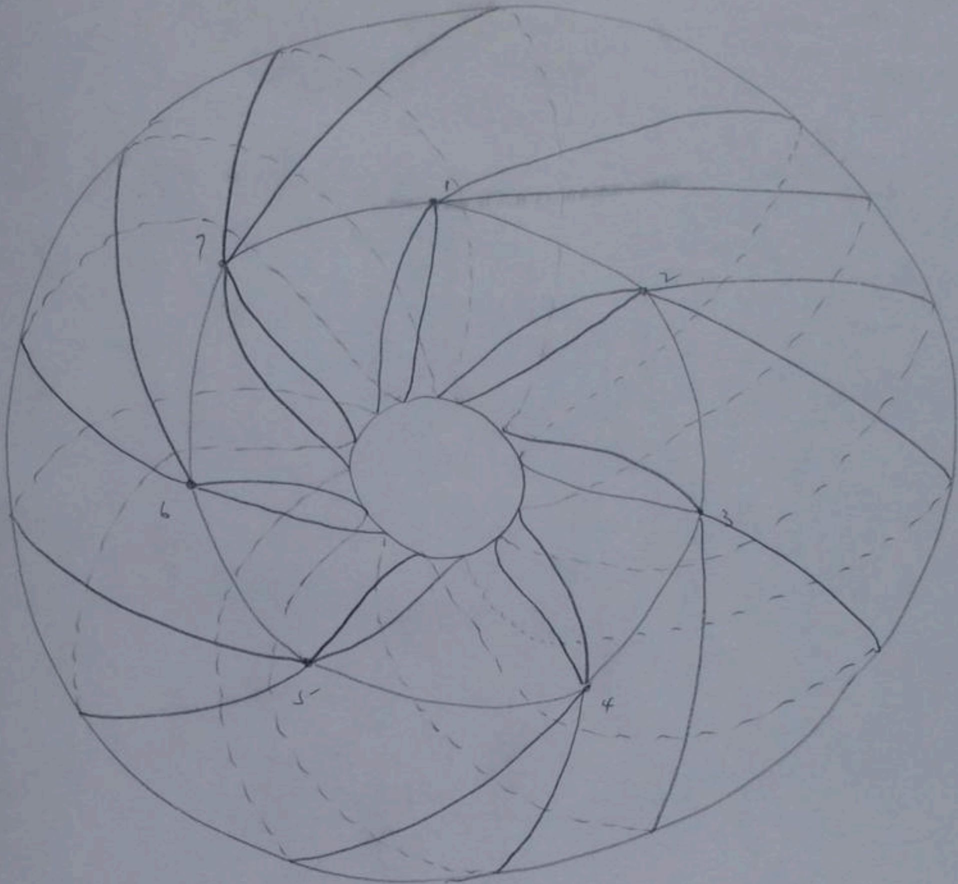
54
56
56
57
54
53





12 − 18 + 6

13	23	32	41	51	61
16	21	34	43	52	62
14	24	35	45	54	63
15	25	36	46	56	65
14	25	31	41	52	63
12	26	36	42	53	64



~~12~~	~~23~~	~~34~~	~~45~~	~~56~~	~~67~~
~~13~~	~~24~~	~~35~~	~~46~~	~~57~~	
~~14~~	~~25~~	~~36~~	~~47~~		
~~15~~	~~26~~	~~37~~			
~~16~~	~~27~~				
~~17~~					

$7 - 14 + 7 = 0.$

$\frac{7 \cdot 4}{2} = 14 = 14 \checkmark$

$x - 21 + 7 = 0$

$x = 14$

An Elementary Fixed Point Theorem and Some Applications. A. Tarski
(Ann Arbor, March 2, 1940)

Theorem 1. Let $f(x)$ real and single-valued be defined for $0 \leqq x \leqq a$ such that $0 \leqq f(x) \leqq a$. If $f(x)$ is monotone increasing, i.e. $x_1 < x_2 \rightarrow f(x_1) \leqq f(x_2)$, then there exists d, $0 \leqq d \leqq a$, such that $f(d) = d$.

Proof. Consider first the simple proof in case $f(x)$ is continuous. Let $g(x) = x - f(x)$. Then $g(0) \leqq 0$ and $g(a) \geqq 0$. But g is continuous. Hence for some d, $0 \leqq d \leqq a$, $g(d) = 0$, i.e. $f(d) = d$.

The general case. Let $S = \underset{x}{E}[x \leqq f(x)]$. Then $S \neq 0$ for $0 \varepsilon S$. Let $d = \underset{x \varepsilon S}{\text{l.u.b.}}\, x$. Now if $x \varepsilon S$, $x \leqq f(x)$ and hence since f is monotone increasing $x \leqq f(x) \leqq f(d)$. Taking l.u.b. we have $d \leqq f(d)$. Now both d and $f(d)$ are values on $0 \leqq S \leqq a$. Hence since f is monotone $f(d) \leqq f[f(d)]$, i.e. $f(d) \varepsilon S$. Hence $f(d) \leqq d$. Having the inequality in both directions we conclude that $d = f(d)$.

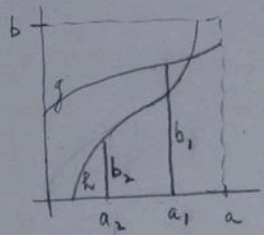
We note that actually more than stated in the theorem has been proved. For let e be any number for which

$0 \leq e \leq a$ and such that $e = f(e)$. Then, by definition $e \in S$. Hence $e \leq d$. Thus d is the 'greatest fixed point' of f on $[0, a]$.

Similarly if we let $T = \underset{x}{E}\,[x \geq f(x)]$ and define $c = \underset{x \in T}{g.l.b.}\, x$, we find that $f(c) = c$ and that c is the 'least fixed point' of f.

<u>Corollary 1.</u> Let $g(x)$ $[h(y)]$ be real, single valued, monotone increasing on $0 \leq x \leq a$ $[0 \leq y \leq b]$ with values on $0 \leq g(x) \leq b$ $[0 \leq h(y) \leq a]$. Then there exist four numbers a_1, a_2, b_1, b_2, $0 \leq a_i \leq a$, $0 \leq b_i \leq b$ such that

$$a = a_1 + a_2 \qquad g(a_1) = b_1$$
$$b = b_1 + b_2 \qquad h(b_2) = a_2$$



Consider $f(x) = h[b - g(a-x)]$. Then $a - x \downarrow$, $g(a-x)\downarrow$, $b - g(a-x) \uparrow$, $h[b - g(a-x)] \uparrow$, $f(x) \uparrow$. And $f(x)$ is defined on $0 \leq x \leq a$ with values on the same interval. Hence by Theorem 1, f has a fixed point, say a_2, $f(a_2) = a_2$

We now define $a_1 = a - a_2$, $b_1 = g(a_1)$, $b_2 = b - b_1$. Then clearly $a_1 + a_2 = a$, $b_1 + b_2 = b$. Further

$$g(a_1) = b_1, \quad h(b_2) = h[b - b_1] = h[b - g(a_1)]$$
$$= h[b - g(a - a_2)] = f(a_2) = a_2.$$

The proof is complete.

If we now analyze the proof of Theorem 1 we find that the only properties of real numbers which we used are order properties. Hence we seek to generalise the theorem to cover more abstract cases.

Consider then an arbitrary class Γ of elements $X, Y, Z, \ldots$, in which there is defined a binary relation $\leqq$ between certain pairs of elements: $X \leqq Y$, etc. We shall consider three postulates.

P_1: If $X, Y \varepsilon \Gamma$, then $X = Y$ if and only if both $X \leqq Y$, $Y \leqq X$.

P_2: If $X, Y, Z \varepsilon \Gamma$, and $X \leqq Y$, $Y \leqq Z$ then $X \leqq Z$.

Before stating P_3 we make the following definitions.

Def. Let $\gamma \subset \Gamma$. We call $D \varepsilon \Gamma$ the l.u.b. (or sum, or union, etc) of γ if D satisfies the following two conditions:

1) $X \varepsilon \gamma \to X \leqq D$

2) If $X \varepsilon \gamma \to X \leqq Y$, then $D \leqq Y$.

It is easily shown that there is at most one l.u.b. for a set γ.

We shall define g.l.b. of a set γ in terms of l.u.b. Let T be the set of all $X \varepsilon \Sigma$ such that if $Y \varepsilon \gamma$, $X \leqq Y$. Then if T has a l.u.b. we define this element to be the _g.l.b._ of γ.

We now state the third postulate.

P_3. Every subset $\gamma \subset \Sigma$ has a l.u.b.

Def. The class Σ is said to be _partially ordered_ if P_1 and P_2 are satisfied in Σ.

_Def._* The class Σ is said to be a _complete_ (or _continuous_) partially ordered set if P_1, P_2 and P_3 are satisfied in Σ.

We come now to the theorem which generalizes Theorem 1.

Theorem 2. _Let_ Σ _be a complete partially ordered set_. _Let_ $F(X)$ _be a single-valued function_

* This is def. of complete lattice. ?

defined over Σ, with values in Σ, and monotone increasing, i.e. such that $X \leq Y \rightarrow F(X) \leq F(Y)$. Then there exists a non-null set γ of elements Z, $\gamma \subset \Sigma$ such that for $Z \varepsilon \gamma$, $F(Z) = Z$, and such that if $F(Y) = Y$, $Y \varepsilon \gamma$. Further γ has maximum and minimum elements, namely l.u.b. γ and g.l.b. γ.

The proof is practically identical with the proof of Th. 1. In place of 0 we have g.l.b. Σ and in place of a we have l.u.b. Σ.

We turn now to an application. One of the simplest complete partially ordered sets is the set of all subsets of a given set A, where by $X \leq Y$ we mean, $X \varepsilon A$, $Y \varepsilon A$, $X \subset Y$. Then l.u.b. is seen to be set theoretic sum and g.l.b. to be set theoretic product. We shall suppose that A lies in a topological space S of the following character. If X is a subset of S certain points of S shall be called limit points of X. The derived set of X, namely X' shall consist not only of the limit points of X but in addition of the limit points of all subsets of X. This definition is seen to be satisfied in more usual topological spaces. We define X to be closed if $X' \subset X$, and to be perfect if $X = X'$. We then have a function defined in Σ the class of all subsets X of A, namely $F(X) = X'$. This function is defined over Σ. If A is closed in S, $X \subset A \rightarrow X' \subset A' = A$, and hence $F(X)$ has values in Σ. Finally $F(X)$ is seen to be monotone for if $X \leqq Y$ then $X' \leqq Y'$. Hence for some $X \varepsilon \Sigma$, $X = F(X) = X'$. We thus have the following corollary of Th. 2.

Cor. 2. *If A is closed in S there exists a greatest perfect subset in A.*

Suppose now that in an arbitrary ordered set Σ there are defined for every pair of elements X and Y, two elements $X+Y$ and XY such that

i) $X \leqq X+Y$ i') $XY \leqq X$

$Y \leqq X+Y$ $XY \leqq Y$

$X \leqq Z, Y \leqq Z \rightarrow X+Y \leqq Z$ $Z \leqq X, Z \leqq Y \rightarrow Z \leqq XY$.

If L is the class of all subsets of A and if we interpret $X+Y$ as set sum and $X \cdot Y$ as set product then i) and i') are satisfied in L.

If in a class L in which P_1 P_2 and i) i') are satisfied L is called a lattice. $X+Y$ and XY are called respectively the l.u.b and g.l.b of X and Y.

Consider a partially ordered class L containing two elements 0 and 1 such that $X \leqq 1$, $0 \leqq X$ for all $X \in L$. [0 and 1 exist in any complete partially ordered set].

We now state two new postulates in which $X+Y$ and XY occur, with it understood that $X+Y$ and XY satisfy i) and i').

P_4: If $X, Y, Z \in L$, $X(Y+Z) = XY + XZ$

P_5: If $X \in L$ $\exists$ $Y \in L \ni X+Y = 1$, $XY = 0$.

Y is called the complement of X and is denoted by $-X$. One can show that if $-X$ exists it is unique.

Def. A lattice satisfying P_4 and P_5 is called a Boolean algebra. A Boolean algebra satisfying P_3 is said to be complete.

X [? If $XY=0$, $X-Y = X(-Y)$ is called the symmetric difference]

Theorem 3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two complete Boolean algebras. Let $A \varepsilon \mathfrak{A}$ and $B \varepsilon \mathfrak{B}$. Let $G(X)$ $[H(Y)]$ be a monotone increasing function defined over $\mathfrak{A}$ $[\mathfrak{B}]$ with values in $\mathfrak{B}$ $[\mathfrak{A}]$, then

$$G(A) \leqq B, \qquad H(B) \leqq A.$$

Then there exist elements $A_1 A_2 \varepsilon \mathfrak{A}$ and $B_1 B_2 \varepsilon \mathfrak{B}$ such that

$$A = A_1 + A_2, \; B = B_1 + B_2, \quad A_1 A_2 = 0, \quad B_1 B_2 = 0$$

and $$G(A_1) = B_1, \qquad H(B_2) = A_2.$$

The proof is identical with the proof of Cor. 1, using $F(X) = H[B - G(A-X)]$

Consider now an arbitrary set A and a function $f(x)$ defined over A. Then f generates a function $f^*(X)$ of subsets of A, defined over Σ the class of subsets of A, i.e.

$$f^*(X) = \underset{f(x)}{E}\left[x \varepsilon X\right]$$

If $f^*(X) = Y$ we say f^* maps X into Y; $X \xrightarrow[f]{} Y$.

We now have the following corollary of Th. 3.

Cor. 3a. Let A and B be two arbitrary sets. Let g and h be arbitrary (single valued) functions defined over A and B respectively such that

$$A \xrightarrow[g]{} B' \subset B, \qquad B \xrightarrow[h]{} A' \subset A.$$

Then $\exists$ $A_1, A_2 \subset A$, $B_1, B_2 \subset B$, such that

$$A = A_1 + A_2, \quad B = B_1 + B_2, \quad A_1 A_2 = B_1 B_2 = 0$$

and $$A_1 \xrightarrow[g]{} B_1 \qquad B_2 \xrightarrow[h]{} A_2.$$

Proof. Consider the complete Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ of all subsets of A and B and the functions g^* and h^* generated by g and h. Then applying Th. 3 we have the corollary.

Suppose now that g and h are biunique. We then have Bernstein's theorem as a

Cor. 3b. If A and B are arbitrary sets such that $\bar{\bar{A}} = \bar{\bar{B'}}$, $(B' \subset B)$, $\bar{\bar{B}} = \bar{\bar{A'}}$, $(A' \subset A)$ then $\bar{\bar{A}} = \bar{\bar{B}}$.

Consider now two sets A and B lying in metric spaces. A mapping f (biuniform) of A on B is said to be <u>isometric</u> if when $x, y \in A$,

$$\rho(x,y) = \rho[f(x), f(y)]$$

If f is isometric and $A \xrightarrow[f]{} B$ we say that A and B are <u>congruent</u>, $A \cong B$. If A and B admit decompositions

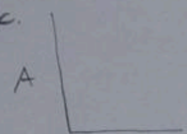
$$A = A_1 + A_2 + \cdots + A_n$$
$$B = B_1 + B_2 + \cdots + B_n$$

such that $A_i \cong B_i$, $i = 1, \ldots, n$ we say that A and B are <u>equivalent</u> by finite decomposition.

We now have the following corollary of Th. 3.

<u>Cor.</u> 3c. <u>If A and B in metric spaces are such that</u> $A \cong B' \subset B$, $B \cong A' \subset A$, <u>then</u> A <u>and</u> B <u>are equivalent</u>.

Ex.



$x \geq 0,\ y \geq 0$

$x \geq 0,\ y \geq 0,\ xy \geq 1$

A and B are not congruent but clearly each is congruent to a subset of the other. The decomposition is indicated at the left.

Lattice satis. P_4 and P_5 – is Boolean alg.

If P_3 satisfied completely additive Boolean alg

$X \cdot Y = 0$

$X - Y = X(-Y)$. Symmetric diff.

Th 3 Let $\mathfrak{A}$ & $\mathfrak{B}$ be two complete B.A.

$A \varepsilon \mathfrak{A}$ $B \varepsilon \mathfrak{B}$. Let $G(X)$ and $H(Y)$ be increasing, G over $\mathfrak{A}$ onto $\mathfrak{B}$, H over $\mathfrak{B}$ onto $\mathfrak{A}$)

$G(A) \leqq B$, $H(B) \leqq A$.

$\exists$ elements $A_1 A_2 \varepsilon \mathfrak{A}$ $B_1 B_2 \varepsilon \mathfrak{B}$

$\Rightarrow A = A_1 + A_2$, $B = B_1 + B_2$, $A_1 A_2 = B_1 B_2 = 0$

$G(A_1) = B_1$, $H(B_2) = A_2$

Def new function $F(X) = H\left[B - G(A - X)\right]$

Then $F\uparrow$ $\therefore$ fixed pt A_2 etc.

Consider A arb. set

f def on A (all over A)

Then f generates a function $f^*(X)$ def. over subsets of A, i.e.

$$f^*(X) = \underset{f(x)}{E}\left[x \varepsilon X\right]$$

If $f^*(X) = Y$ we say f^* maps X onto Y.

$$X \xrightarrow[f]{} Y$$

If f has prop. that $f(x) \neq f(y)$ $x \neq y$ we have following cor. 3c. Let A and B be two arb. sets and g, h arb. function g over A, h over B. $A \xrightarrow[g]{} B' \subset B$

$$B \xrightarrow[h]{} A' \subset A$$

Applications.

Ex. A arb. set. Γ class of all subsets of A is complete partially ordered if $X \leq Y \leftrightarrow X \subset Y$. l.u.b. = set theoretic sum

Assume A lies in topol. space, very general character:

[S top. space, $X \subset S$, $p \in S$

X' = set of limit pts of X and of all limit pts of all subsets of X.

X closed if $X' \subset X$

X perfect if $X' = X$

Derivative is increasing function

If $X \leq Y$, $X' \leq Y'$

∴ Cor. of Th. 2) If A is closed subset of S

∃ greatest perfect subset in A

from P_3 that any subclass Γ' of Γ has not only l.u.b. but g.l.b. For:

let $\Gamma' \subset \Gamma$, Consider $\overset{T}{[X \in \Gamma]} \ni$ if $Y \in \Gamma'$, $X \leq Y$.

Then T by P_3 has l.u.b., and is g.l.b. of Γ'.

In part. Γ has l.u.b. dem. by 1, $X \leq 1$

g.l.b. " ; 0, $0 \leq X$

Any subclass of Γ, containing two elements X, Y with l.u.b. $= X + Y$, g.l.b. $= XY$

∴ Every complete p. ord. set is ~~complete~~ lattice

Add

P_4 $X, Y, Z \in \Gamma$, $X(Y+Z) = XY + XZ$

P_5 $X \in \Gamma$ ∃ $Y \in \Gamma \ni X + Y = 1$
$X \cdot Y = 0$

Y is called complement. Can show unique

Denote by $-X$

3

Consider

$$f(x) = h\left[b - g(a-x)\right]$$

Then $f(x) \uparrow$ def. on $[0,a]$ values on $[0,a]$

$\therefore f(x)$ has fixed pt. a_2, $f(a_2) = a_2$

Then def. $a_1 = a - a_2$

$b_1 = g(a_1)$

$b_2 = b - b_1$

Analyze now proof of Th. Only very few properties of real numbers used – all props. are order props. $\therefore$ generalize

Consider ab. class Σ of elements $X, Y, Z \ldots$

binary rel. $\leqq$, $x \leqq y$ + ∋

P_1: $X, Y \in \Sigma$, $X = Y \Leftrightarrow X \leqq Y, Y \leqq X$

P_2: $X, Y, Z \in \Sigma$, $X \leqq Y, Y \leqq Z \rightarrow X \leqq Z$

~~Def~~ If $\mathcal{Y} \subset \Sigma$ we call $D \in \Sigma$ l.u.b. (sum, union) of $\mathcal{Y}$

1) $X \in \mathcal{Y}$ $X \leqq D$

4

$\forall \left(X \in \mathcal{Y} \rightarrow X \leqq Y\right)$

$\rightarrow D \leqq Y$

P_3: Any $\mathcal{Y} \subset \Sigma$ has a l.u.b. D of $\mathcal{Y}$

Def. Σ part. ordered if $P_1\ P_2$

Def. ~~If~~ $P_1\ P_2\ P_3$ Σ complete or contin. partially ordered set.

Th.2 Let Σ be a complete p. or. set.

Let $F(X)$ be $\uparrow$ def. over Σ with values in Σ

Then $F(X)$ has fixed pt. i.e. $\exists\, D \in \Sigma$ ∋ $F(D) = D$ and the set of all fixed pts has max + min.

$f(x)$ nondim $[0,a]$ values in same int.

Then $f(x)$ has fixed pt. $f(d)=d$

$S = \underset{x}{E}[x \leq f(x)]$ Then $S \neq 0$, for $0 \in S$.

S has l.u.b. Let $d = $ l.u.b S.

Then show $f(d)=d$

Let $x \in S$ then $x \leq f(x)$

$x \leq d$

Now $f(x) \uparrow \therefore f(x) \leq f(d) \therefore x \leq f(d)$

$\therefore d \leq f(d)$

But $f \uparrow \therefore f(d) \leq f(f(d))$

$\therefore f(d) \in S, \therefore f(d) \leq d$

$\therefore f(d) = d$ ✓

d is not merely fixed pt, but is greatest fixed pts, since all fixed pts $\in S$,

Set of all fixed pts not empty but has a max.

Similarly has min.

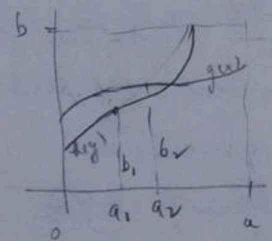
Use $T = \underset{x}{E}[x \geq f(x)]$

$c = $ g.l.b. Then c is fixed & min.

Cor. Let $g(x) \uparrow$ def over $0 \leq x \leq a$

$0 \leq g(x) \leq b$

$h(y) \uparrow \quad 0 \leq y \leq b, \quad 0 \leq h(y) \leq a.$



$\exists \quad a_1, a_2, \quad b_1, b_2 \Rightarrow \quad a = a_1 + a_2$

$b = b_1 + b_2$

$g(a_1) = b_1$

$h(b_2) = a_2$

Then $A = A_1 + A_2$, $B = B_1 + B_2$ $A_1 A_2 = B_1 B_2 = 0$

$\Rightarrow$ $A_1 \xrightarrow[g]{} B_1$, $B_2 \xrightarrow[h]{} A_2$

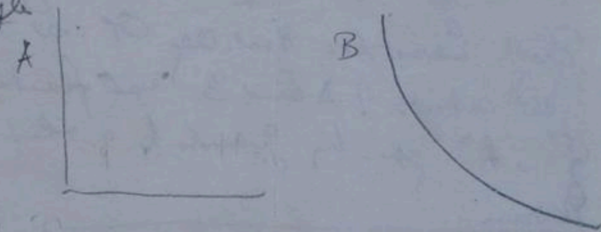
Proof. Consider Bool. Alg. $\mathfrak{A}$ & $\mathfrak{B}$
all subsets of A and B and function
g^*, h^* gen. by g and h & then apply
Theorem

Suppose now g and h are arbitrary
biunique function
Cor. If A and B sets with
$\overline{A} = \overline{\overline{B'}}$, $\overline{\overline{B}} = \overline{\overline{A'}}$. (Bernstein)

Let A and B lie in metric spaces. Then
f is called isometric if def such that
$\rho(x, y) = \rho(f(x), f(y))$.
If A and $B \ni$ $A \xrightarrow[f]{} B$ f isometric.
We call A and B congruent.
$A \cong B$ if $A = A_1 + \cdots + A_n$ $A_i \cong B_n$
$B = B_1 + \cdots + B_n$

Cor. If A and B in metric space $A \cong B'$, $B \cong A'$ then $A \cong B$.

<u>Example</u>

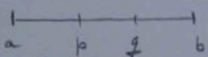


A and B not cong., but ~~not~~ equiv.

∴ can be decomp. into 2 cong. parts.

Example of Metric Space containing a set E such that $\delta(E)$ is not the greatest lower bound of diameters of spheres containing E.

Space consists of points of segment of ordinary length 1. Denote endpoints by a and b. Denote by p and q any two points of segment distinct from a and b. Let $|ap|$ be the ordinary distance from a to p. Similarly let $|bp|$ be the ordinary distance from b to p. Let xy denote the metric distance to be defined.



Def.

$$ab = 2$$
$$ap = 2 + |ap|$$
$$bp = 2 + |bp|$$
$$pq = 3, \quad p \neq q$$
$$xy = yx, \quad xx = 0.$$

The metric thus defined is symmetric, and vanishes only if the points are identical.

Verification of triangle inequality.

If x, y, z are all distinct from a and b, the inequality is obviously satisfied. The only other essentially different cases, because of the symmetry of a and b, are the following:

p. q. a. $\quad pq \leq pa + qa, \quad$ for $\quad 3 \leq 2 + |pa| + 2 + |qa|$

$\quad pa \leq pq + aq, \quad$ for $\quad pa < 3 < 3 + 2 + |aq|$

p. a. b. $\quad pa \leq pb + ab, \quad$ for $\quad pa < 3 < 2 + |pb| + 2$

$\quad ab \leq pa + pb, \quad$ for $\quad 2 < 2 + |pa| + 2 + |pb|$

Let now E be the set (a, b).

Consider any sphere $K(x, r)$ containing E. There are just two types of choices for x.

1) $x = a$. Then $r > 2$, say $r = 2 + \epsilon$

Let p and q be any points such that $|ap| < \epsilon$, $|aq| < \epsilon$.

Then $p \in K$ and $q \in K$. Hence the diameter of $K = 3$.

2) $x = p$.

Suppose, for definiteness that $|ap| \leq \frac{1}{2}$. Then $|bp| \geq \frac{1}{2}$.

Hence r must be $> 2\frac{1}{2}$. Then K contains a, b <u>and</u> p.

Hence diameter of $K = bp \geq 2\frac{1}{2}$.

Hence the lower bound of diameters of spheres containing E is $2\frac{1}{2}$. But the diameter of E is 2.